Belief Propagation for Optimal User Association in Wireless Networks

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Research Project

May 10, 2012

1 Introduction

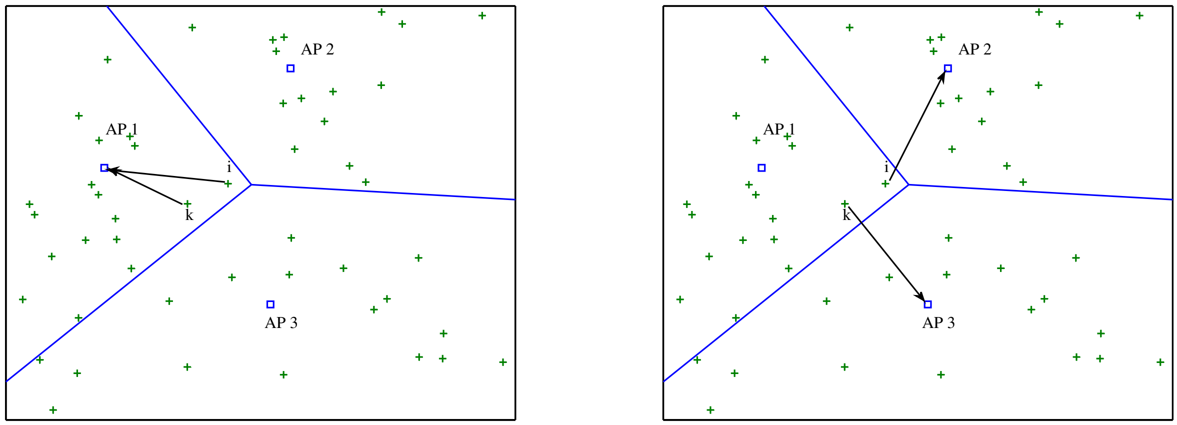
In a wireless setting the effective rate delivered to a user is not only influenced by the spectral efficiency of its link with the associated access point (AP) but also by the total number of users associated with that particular AP (also called the load). Thus, any association based on local greedy decisions by the user would tend to be suboptimal. For example, a greedy association policy where each user connects to its strongest (in terms of spectral efficiency) AP may result in all users connecting to the same AP driving down the effective rate of each user. The optimization formulation for user association is hard due to its combinatorial nature. In this report, we investigate low complexity belief propagation methods to achieve optimal user association, which are a naturally attractive option due to already existent messaging protocols in wireless networks.

1.1 Related Work

There has been limited work in the application of belief propagation (BP) algorithms in wireless settings. BP was used in [1] to optimize the selection of MIMO beamforming vectors so as to maximize system sum rate. In [2] belief propagation was used for inter-cell interference coordination with the objective of maximizing a certain network utility. However, in all these works the user associations with the APs was assumed to be fixed (given) and system was optimized with respect to resource allocation. Analogy of the user association problem can be drawn with the clustering problem considered in the literature [3] where the objective is to cluster similar "looking" data points so as to minimize the mean squared error. However, the clustering problem does not extend naturally to a network setting where the network utility is also influenced by the number of users clustered to a data point. We plan to explore this further in future investigations.

1.2 Contribution

In this report, we formulate the optimal user association as a network utility maximization problem. The solution of the optimal user association is posed as the MAP estimate of suitably constructed probability distribution. A max-product message passing algorithm is presented to obtain the MAP estimate. Through numerical results, it is shown that the exact max product is significantly better than greedy association and is close to optimal. However, message product algorithm still suffers from high complexity. To counter this, we further propose an approximate



(a) (b)

Figure 1: Users i and k associated with AP1 under greedy association (a). User i associated with AP2 and user k associated with AP3 under optimal association (b). Plus mark denotes a user and a square box denotes an AP

low complexity BP algorithm which is shown to perform better than the greedy association in scenarios with large number of users, which is typically the case in cellular wireless networks.

2 Problem Formulation

2.1 System Model

A typical wireless setting is shown in Fig. 1 where a set of users lying in the 2D plane want to associate to one of the available APs, (AP1, AP2, AP3, say). Let N_{tx} denote the set of all the APs and N_{rx} denote the set of all the users in the system. Each link between each user, i and AP, j is associated with a Signal-to-Interference-plus-Noise-Ratio, SINR_{ij}. The set of all such links is denoted by E. Further let E_k denote the set of all links that are incident on node k. SINR of each link is assumed to be known at both the corresponding user and the AP.

Every AP $\in \mathrm{N}_{\mathrm{tx}}$ is assumed to be having a fixed resource B. Note that the resource can be interpreted as bandwidth or time frequency blocks for OFDMA systems. The spectral efficiency of a user i associated with AP j is assumed to be given by the Shannon's channel capacity formula :

$$S_{i,j} = \log(1 + \mathrm{SINR}_{i,j}).$$

Assumption 1. *We assume fair resource allocation by each AP in which each user gets equal fraction of resources. This assumption is clearly valid for round robin resource allocation but for scheduling techniques like proportional fair (PF) this is also shown to hold true in the equilibrium state of PF [4].*

Thus, if K users are connected to AP, j, the effective rate of user i associated with j is given by

$$R_{i,j} = \frac{B \times S_{i,j}}{K} = \frac{B \log(1 + \mathrm{SINR}_{i,j})}{K}. \tag{1}$$

2.2 User association optimization problem

Let the variable $x_{i,j}$ be the indicator function of the event that user i is associated with AP j.

Assumption 2. *It is assumed that a user can associate with at most and at least one AP.*

Thus, under the above assumption the following identity is evident :

$$\sum_{j \in \mathrm{N_{tx}}} x_{i,j} = 1 \, \forall i \in \mathrm{N_{rx}} \tag{2}$$

Thus, the user association problem can be formulated as,

$$\begin{aligned} &\max \sum_{j \in \mathrm{N_{tx}}} \sum_{i \in \mathrm{N_{rx}}} x_{i,j} U(R_{i,j}) \\ &\text{s.t} \sum_{j \in \mathrm{N_{tx}}} x_{i,j} = 1 \, \forall i \in \mathrm{N_{rx}}; x_{i,j} \in \{0,1\}. \end{aligned} \tag{3}$$

where $U(R)$ is some utility function of rate. For example, if $U(R) = R$, then (3) corresponds to sum rate maximization and user association problem can be written as,

$$\begin{aligned} &\max \sum_{j \in \mathrm{N_{tx}}} \sum_{i \in \mathrm{N_{rx}}} \frac{x_{i,j} S_{i,j} B}{\sum_{i \in \mathrm{N_{rx}}} x_{i,j}} \\ &\text{s.t} \sum_{j \in \mathrm{N_{tx}}} x_{i,j} = 1 \, \forall i \in \mathrm{N_{rx}}; x_{i,j} \in \{0,1\}. \end{aligned} \tag{4}$$

3 Belief propagation for optimal user association

3.1 Exact Max-Product

When a probability distribution can be expressed as a product of factors, each of which depends only on a subset of variables, the Max-product form of belief propagation can be used to find the most likely state, the MAP estimate, of the corresponding probability distribution. Max-product operates by iteratively passing messages between variables and the factors that they are part of. In order to apply max-product, we now formulate user association as a MAP estimation problem, by constructing a suitable probability distribution. Hereafter, binary variable x_e associated with each edge $e \in E$ is equivalent to $x_{i,j}$. Consider the following probability distribution:

$$p(\mathbf{x}) \propto \prod_{i \in N_{rx}} \Psi_i(x_{Ei}) \prod_{j \in N_{tx}} \Phi_j(x_{Ej}) \tag{5}$$

which contains a factor

$$\Psi_i(x_{Ei}) = 1 \left(\sum_{e \in Ei} x_e = 1 \right) \tag{6}$$

for each user $i \in \mathrm{N_{rx}}$ and a factor

$$\Phi_j(x_{Ej}) = \exp \left(u \sum_{e \in E_j} x_e U(R_e) \right) \tag{7}$$

for each AP $j \in \mathrm{N}_{\mathrm{tx}}$ where u is some constant. Thus, each variable signifying an edge is part of exactly two factors, each corresponding to one of its end point. Looking at p it is clear that the mode of the distribution maximizes the sum utility when the constraint of (3) imposed by the indicator function is valid. The factor-graph version of the max-product algorithm [5] passes messages between variables and the factors that contain them at each iteration. The output is an estimate $\hat{x}$ of the MAP of p. The max-product update equations for the p in (5) are given in Algo. 1. Algo. 1 can be partly simplified by noting the factor to variable update equation

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Algorithm 1 Max-Product for User Association

- **INIT** Set t=0 and initialize each message to 1
- **ITER** Iteratively compute new messages until convergence as follows:
  Variable to Factor
  $$m_{e\to i}^{t+1}(x_e) = m_{j\to e}^{t}(x_e)\ \forall e \equiv (i,j)$$
  $$m_{e\to j}^{t+1}(x_e) = m_{i\to e}^{t}(x_e)\ \forall e \equiv (i,j)$$
  Factor to Variable :
  $$m_{i\to e}^{t+1}(x_e) = \max_{x_{Ei}\setminus e}\left\{\Psi_i(x_{Ei})\prod_{e'\in Ei\setminus e} m_{e'\to i}^{t}(x_{e'})\right\}\ \forall\ i \in \mathrm{N}_{\mathrm{rx}}$$
  $$m_{j\to e}^{t+1}(x_e) = \max_{x_{Ej}\setminus e}\left\{\Phi_j(x_{Ej})\prod_{e'\in Ej\setminus e} m_{e'\to j}^{t}(x_{e'})\right\}\ \forall\ j \in \mathrm{N}_{\mathrm{tx}}$$
  At each time t compute beliefs,
  $$b_e^t(x_e) = m_{i\to e}^{t}(x_e) \times m_{j\to e}^{t}(x_e)$$
- **ESTIM** Each edge e has estimate
  $$\hat{x}_e^t = 1 \text{ if } b_e^t(1) > b_e^t(0)$$
  $$\hat{x}_e^t = 0 \text{ if } b_e^t(0) > b_e^t(1)$$
  $$\hat{x}_e^t = x_{\mathrm{greedy}} \text{ if } b_e^t(1) = b_e^t(0)$$
  where x_{greedy} is the solution of the greedy association.

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and (6). The simplified steps are given in Algo. 2.

3.2 Gaussian approximation

However, even after the presented simplification the max-product algorithm presented suffers from high complexity. Specifically, the **ITER** step in Algo. 1 still requires search over $2^{\mathrm{N}_{\mathrm{rx}}}$ states in each iteration. This search becomes prohibitive as N_{rx} grows. Thus, to simplify the computational complexity we relook at the MAP estimation algorithm. Instead of using max product algorithm we use the sum product algorithm for estimating marginals of the distribution p. Using the standard large deviation result [6] that with $u \to \infty$, p concentrates

Algorithm 2 Simplification of **ITER** of Algorithm 1

Factor to Variable :

$$m^{t+1}_{i\to e}(x_e) = \max_{x_{Ei}\backslash e}\left\{\Psi_i(x_{Ei})\prod_{e'\in Ei\backslash e} m^t_{e'\to i}(x_{e'})\right\} = \prod_{e'\in Ei\backslash e} m^t_{e'\to i}(0);\ \text{if } x_e = 1$$

$$\begin{aligned} m^{t+1}_{i\to e}(x_e) &= \max_{x_{Ei}\backslash e}\left\{\Psi_i(x_{Ei})\prod_{e'\in Ei\backslash e} m^t_{e'\to i}(x_{e'})\right\} \\ &= \prod_{e'\in Ei\backslash e} m^t_{e'\to i}(0)\max_{e'\in E_i\backslash e}\frac{m_{e'\to i}(1)}{m_{e'\to i}(0)};\ \text{if } x_e = 0 \end{aligned}$$

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around its mode, we can estimate the marginals of p for large u and can recover a good estimate for the maximization of (3). The corresponding sum product algorithm is given in Algo. 3. The reformulation as Algo. 3 does not solve the computational complexity problem as (9) still

Algorithm 3 Sum-Product for User Association

- **INIT** Set t=0 and let $p_{i\to e}(t, x_e)$ be an uniform initial distribution on $x_e \forall e \in E$.
- **ITER** Iteratively compute new messages as follows:
  Variable to Factor

  $$p^{t+1}_{e\to i}(x_e) = \frac{1}{Z}p^t_{j\to e}(x_e)\ \forall e \equiv (i,j)$$

  $$p^{t+1}_{e\to j}(x_e) = \frac{1}{Z}p^t_{i\to e}(x_e)\ \forall e \equiv (i,j)$$

  Factor to Variable :

  $$p^{t+1}_{i\to e}(x_e) = \mathbb{E}\left[\Psi_i(x_{Ei})|x_e\right]\ \forall\ i \in \mathrm{N_{rx}} \tag{8}$$

  $$p^{t+1}_{j\to e}(x_e) = \mathbb{E}\left[\Phi_j(x_{Ej})|x_e\right]\ \forall\ j \in \mathrm{N_{tx}} \tag{9}$$

  where the expectation is over independent $x_e \sim p_{e\to i}$ in (8) and over independent $x_e \sim p_{e\to j}$ in (9). At each time t compute beliefs as,

  $$b^t_e(x_e) = p^t_{i\to e}(x_e) \times p^t_{j\to e}(x_e)$$

- **ESTIM** Same as Algo 1

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requires sum over $2^{\mathrm{N_{rx}}}$ states. We take the following assumption to simplify the analysis.

Assumption 3. *The sum of variables $x_c, c \in E$ for some set E with large $|E|$ is assumed to be a Gaussian random variable as,*

$$\sum_{c\in E_j} x_c = Y \sim \mathcal{N}(\mu_j, \sigma_j^2), \tag{10}$$

where $\mu_j = \sum_{c\in E_j} p_{c\to j}(1)$ *and* $\sigma_j^2 = \sum_{c\in E_j} p_{c\to j}(1)p_{c\to j}(0)$.

Using the above assumption the factor to variable update in (9) can simplified as below for two utilities:

- $U(R) = R$

$$\mathbb{E}\left[\Phi_j(x_{Ej})|x_e\right] = \mathbb{E}\left[\exp\left(u\frac{x_e\log(1+\mathrm{SINR}_e)+\sum_{c\in E_j/e}x_c\log(1+\mathrm{SINR}_c)}{x_e+\sum_{c\in E_j/e}x_c}\right)|x_e\right] \quad (11)$$

$$= \int \exp\left(u\frac{x_e\log(1+\mathrm{SINR}_e)+y\bar{w}(E_j/e)}{y+x_e}\right)\exp\left(-\frac{(y-\mu_j(e))^2}{2\sigma_j^2(e)}\right)\mathrm{d}y, \quad (12)$$

where $\mu_j(e) = \sum_{c\in E_j/e} p_{c\to j}(1)$ and $\sigma_j^2(e) = \sum_{c\in E_j/e} p_{c\to j}(1)p_{c\to j}(0)$ The following further simplification is used:

$$\sum_{c\in E_j/e} x_c\log(1+\mathrm{SINR}_c) \approx \bar{w}(E_j/e)\sum_{c\in E_j/e} x_c = \bar{w}(E_j/e)Y$$

with

$$\bar{w}(E_j/e) = \frac{1}{|E_j|-1}\sum_{c\in E_j/e}\log(1+\mathrm{SINR}_c).$$

- $U(R) = \log(R)$

$$\mathbb{E}\left[\Phi_j(x_{Ej})|x_e\right] = \mathbb{E}\left[\exp\left(u\left\{x_e\log\left(\frac{\log(1+\mathrm{SINR}_e)}{x_e+\sum_{c\in E_j/e}x_c}\right)+\sum_{c\in E_j/e}x_c\log\left(\frac{\log(1+\mathrm{SINR}_c)}{x_e+\sum_{c\in E_j/e}x_c}\right)\right\}\right)|x_e\right] \quad (13)$$

$$= \int \exp\left(u\{x_e\log(\log(1+\mathrm{SINR}_e))+y\bar{w}(E_j/e)-x_e\log(y+x_e)-y\log(y+x_e)\}-\frac{(y-\mu_j(e))^2}{2\sigma_j^2(e)}\right)\mathrm{d}y, \quad (14)$$

where the following simplifying assumption is used

$$\sum_{c\in E_j/e} x_c\log(\log(1+\mathrm{SINR}_c)) \approx \bar{w}(E_j/e)\sum_{c\in E_j/e} x_c = \bar{w}(E_j/e)Y$$

with

$$\bar{w}(E_j/e) = \frac{1}{|E_j|-1}\sum_{c\in E_j/e}\log(\log(1+\mathrm{SINR}_c)).$$

Thus, with the above presented simplifications the sum over $2^{\mathrm{N_{rx}}}$ states in (9) is converted to a single integral. Table 1 compares the computational complexity of the two proposed algorithms.

Table 1: Computational Complexity per Round of per AP or User

Method	**RX**	**TX**
Exact BP	$\mathcal{O}(\mathrm{N_{tx}})$	$\mathcal{O}(2^{\mathrm{N_{rx}}})$
Gaussian Approximation	$\mathcal{O}(\mathrm{N_{tx}})$	$\mathcal{O}(\mathrm{N_{rx}})$

3.3 Implementation of message passing in wireless networks

So far we have derived algorithms where the messages are passed between variables and factor nodes where variables are the indicator functions of the each possible link in the network. This formulation can not be directly implemented. However, the message passing algorithms can be molded as follows to make them amenable for implementation :

- **Variable to Factor**: Each user calculates $p^{t+1}_{e\to i}(x_e)$ using the unicast message, $p^{t}_{j\to e}(x_e)$, from the corresonding AP in the last slot. Each AP calculates $p^{t+1}_{e\to j}(x_e)$ using the unicast message, $p^{t}_{i\to e}(x_e)$, from the corresponding user in the last slot.
- **Factor to Variable**: Each user calculates $p^{t+1}_{i\to e}(x_e)$ using $p^{t}_{e\to i}(x_e)$ and transmits it to the corresponding AP. Each AP calculates $p^{t+1}_{j\to e}(x_e)$ using $p^{t}_{e\to j}(x_e)$ and transmits it to the corresponding user.
- After a certain number of rounds, each user can estimate the belief and associate with the thus estimated AP.

4 Numerical Results

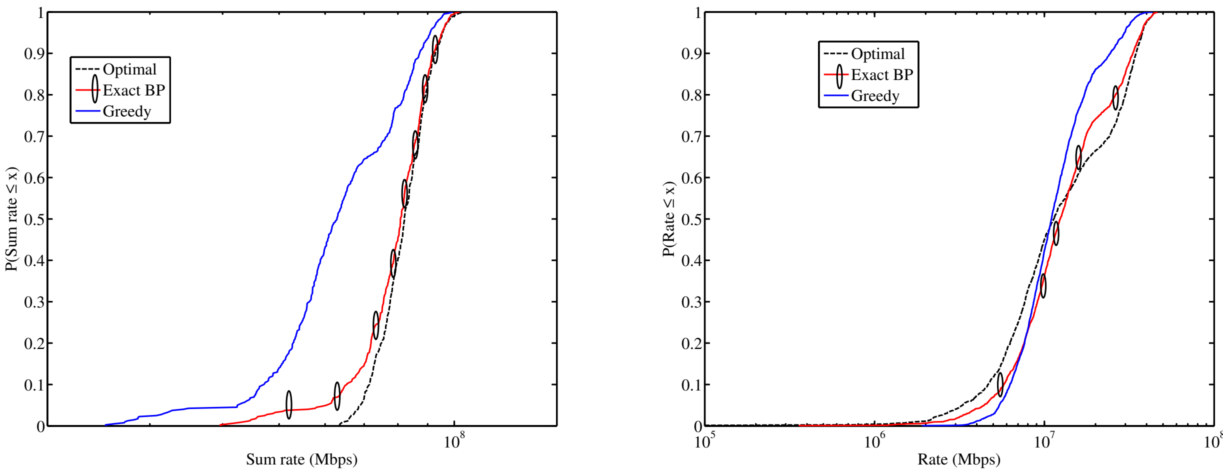
4.1 Exact BP

The performance of exact BP presented in Sec 3.1 is analyzed first in a simplistic setting. The plots for the optimal association are generated through exhaustive search over $({\mathrm{N_{tx}}}^{\mathrm{N_{rx}}})$ choices. In the greedy association each user associates with the AP offering the best spectral efficiency. For the performance comparison of these algorithms, we consider a network with $\mathrm{N_{rx}} = 5$, $\mathrm{N_{tx}} = 3$. The simulation is done for 500 drops of the user and for each drop random spectral efficiencies are generated as $\mathrm{SINR}_{i,j} \sim 5 \times \exp(\lambda_j)$ where $\lambda_j = 1; 2; 3$ for $j = 1; 2; 3$ respectively.

First $U(R) = R$ is considered and thus the objective is sum rate maximization across all users. Fig. 2a shows the CDF of sum rate of the network and Fig. 2b shows the CDF of rate across all users. As seen from the plots, the Algo. 1 is better than the greedy association and is close to optimal for certain regimes. Since sum rate maximization tends to be unfair, it is observed that in Fig. 2b optimal association tends to be worse for users receiving low rate than the other algorithms. But this problem is alleviated if the objective function is changed to $U(R) = \log(R)$ which brings in the fairness across users. Fig. 3b shows that optimal is considerably better in improving the rate of worst users. Again, as can be observed from Fig. 3a and Fig. 3b exact BP (Algo. 1) is better than the greedy association.

4.2 Approximate belief propagation

As mentioned earlier optimal user association for large $\mathrm{N_{rx}}$, which is typically the case in a wireless setting, is highly computationally intensive. We thus compare the performance of approximate belief propagation proposed in Sec. 3.2 with greedy association for $\mathrm{N_{rx}} = 100$ and $\mathrm{N_{tx}} = 3$. The simulation is done for 500 drops of the user and for each drop random



(a) (b)

Figure 2: Cumulative distribution function of the sum rate (a) across network and rate (b) across 5 users with the sum rate maximization utility under different association algorithms.

spectral efficiencies are generated as $\mathrm{SINR}_{i,j} \sim 5 \times \exp(\lambda_j)$ where $\lambda_j = 1; 0.2; 5$ for $j = 1; 2; 3$ respectively.

For $U(R) = R$, Fig. 4a and 4b show the CDF of sum rate and rate across all users respectively. As seen, approximate BP is still better than the greedy association in terms of sum rate and rate in most of the regimes. However for very low rate users, approximate BP tends to be a little worse than the greedy association. Again, this is due to the inherent unfairness in the utility. This effect is ameliorated with $U(R) = \log(R)$. The corresponding CDF are shown in Fig. 5a and 5b where the objective of the optimization is sum log rate maximization. Again as can be seen from the plots, the approximate BP outperforms the greedy association. Note that the exact BP and optimal (exhaustive search) based association results could not be furnished in these plots due to prohibitively large search space.

5 Conclusion

Two belief propagation algorithms : exact and approximate are proposed for optimal user association in wireless networks. The approximate BP is computationally less expensive compared to exact BP and provides superior performance compared to greedy associations. The future work should explore the connections between our work and the AMP framework [7] to obtain performance bounds for the algorithms presented here. Future work should also further explore the connections of the presented optimal user association problem to the affinity propagation problem for data clustering [3].

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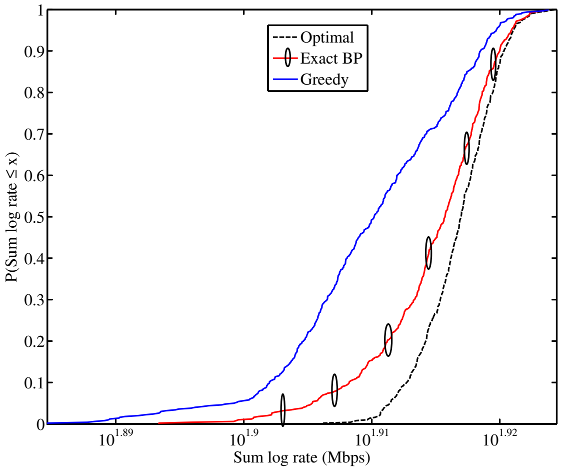
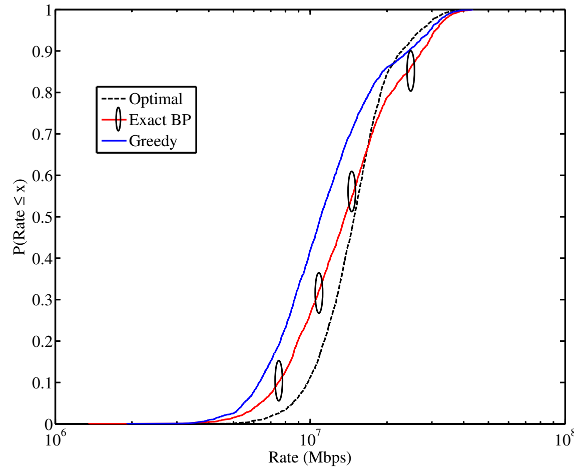
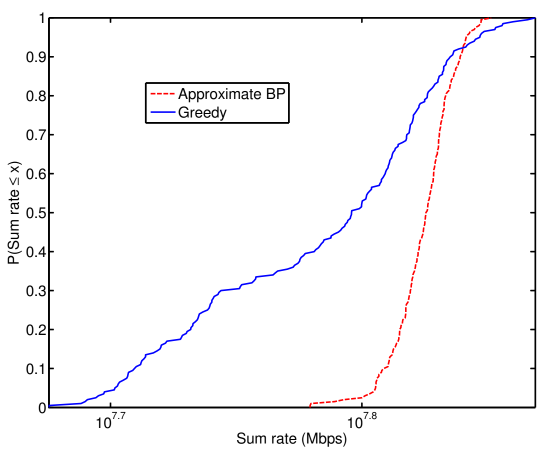


(a)

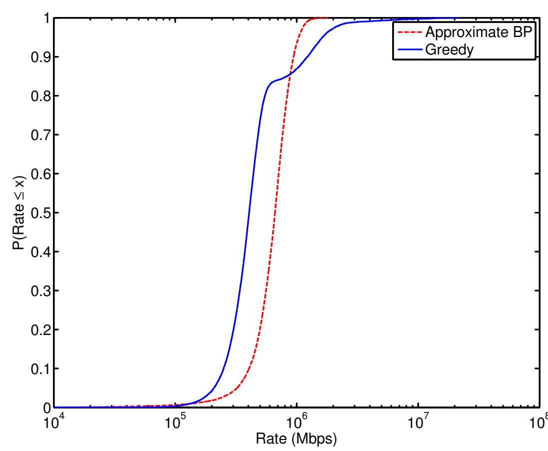


(b)

Figure 3: Cumulative distribution function of the sum rate (a) across network and rate (b) across 5 users with the sum log rate maximization utility under different association algorithms

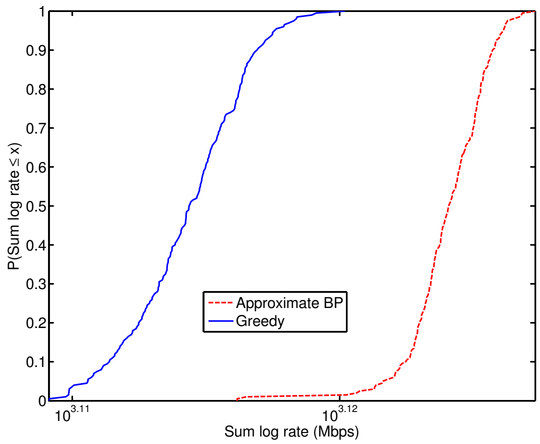


(a)

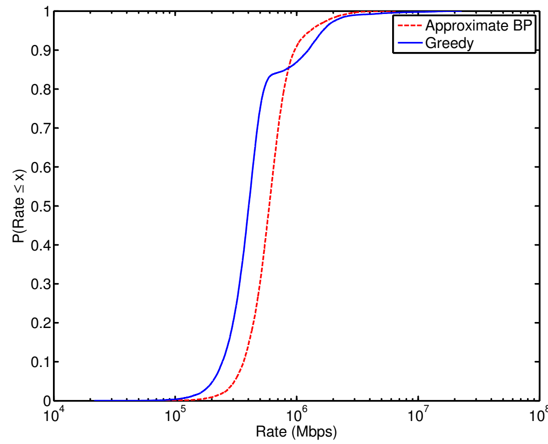


(b)

Figure 4: Cumulative distribution function of the sum rate (a) across network and rate (b) across 100 users with the sum rate maximization utility under different association algorithms



(a)



(b)

Figure 5: Cumulative distribution function of the sum rate (a) across network and rate (b) across 100 users with the sum rate maximization utility under different association algorithms

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